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# CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

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ABSTRACT. We study a general theory of mixed-type operators containing the Tricomi operators, degenerate hyperbolic operators, and elliptic operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

Let $P(x,D)$ be a microdifferential operator defined at $x^*=(0;0,\cdots,0,\sqrt{-1})$ $\in\sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ of order $m\geq 2$, written in the form

$$(1)\qquad \begin{cases} P(x,D)=D_1^m+\sum_{0\leq j\leq m-1} P_j(x,D')D_1^j, \\ \operatorname{ord} P_j\leq m-j. \end{cases}$$

Here we have written $D'=(D_2,\cdots,D_n)$. We also write as $D''=(D_1,\cdots,D_{n-1})$, $D'''=(D_2,\cdots,D_{n-1})$. Let $\sigma_m(P)(x,\xi)$ be the principal symbol of $P(x,D)$. We assume that

$$(2)\qquad \begin{cases} \text{if } x_1=0, \text{ then } \sigma_m(P)=\xi_n^m; \\ \text{if } x_1\neq 0, \text{ then the equation } \sigma_m(P)=0 \text{ has } m \text{ distinct roots} \\ \xi_1=\varphi_1(x,\xi'),\cdots,\varphi_m(x,\xi'). \end{cases}$$

We denote by $\mathcal{O}$ (resp. $\mathcal{O}_{(j)}$) the sheaf of holomorphic functions (resp. the sheaf of functions $f(x_1^{1/j},x')$ such that $f(x)$ are holomorphic). Without loss of generality, we may assume that $\varphi_j(x,\xi')\in\mathcal{O}_{(m'),x^*}$ for some $m'\in\mathbf{N}$, that they are homogeneous in $\xi'$ of degree 1, and that they vanish when $x_1=0$. From now on, we denote $\bar{\mathcal{O}}=\mathcal{O}_{(m')}$. It follows that

$$\begin{cases} \text{for some } q_j\in\mathbf{N}/m' \text{ and some } a_j(x,\xi')\in\bar{\mathcal{O}}_{x^*} \text{ we have} \\ \varphi_j(x,\xi')=x_1^{q_j}a_j(x,\xi'),\ a_j(x^*)\neq 0\ (1\leq j\leq m). \end{cases}$$

We also assume that

$$(3)\qquad i\neq j \quad\Longrightarrow\quad (q_i,a_i(x^*))\neq(q_j,a_j(x^*)).$$

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We denote by $\mathcal{C}$ (resp. $\mathcal{E}$) the sheaf of microfunctions (resp. microdifferential operators). Let us consider the Cauchy problem

$$\begin{cases} Pu = 0, \\ D_1^{j-1}u(0,x') = v_j(x'), \quad 1 \leq j \leq m, \end{cases} \tag{4}$$

where $u \in \mathcal{C}_{\mathbf{R}^n,x^*}$ and $v_j \in \mathcal{C}_{\mathbf{R}^{n-1},x^{*\prime}}$ $(x^{*\prime} = (0;0,\cdots,0,\sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1})$. If $P(x,D)$ is microhyperbolic, (4) is well-posed for arbitrary initial values, as is well-known (See [3]). Otherwise (4) may be solvable for some initial values (e.g., for $v_1 = \cdots = v_m = 0$), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (4) becomes solvable.

To give the main theorem we need to prepare some preliminaries. Let $A(x',D')$ be an both-side invertible $m \times m$ matrix whose components $A_{(\mu,\nu)}(x',D') \in \mathcal{E}^{\mathbf{R}}_{x^*}$ are independent of $(x_1,D_1)$. Here we denote by $\mathcal{E}^{\mathbf{R}}$ the sheaf of holomorphic microlocal operators (c.f. [1,6]). We choose $r$ rows of this matrix in an arbitrary way. To be clear, let $1 \leq j_1 < j_2 < \cdots < j_r \leq m$ and choose the $j_1,\cdots,j_r$-th rows of $A$. Then we obtain an $r \times m$ matrix $A'(x',D')$ of holomorphic microlocal operators. We say that $v_1(x'),\cdots,v_m(x') \in \mathcal{C}_{\mathbf{R}^{n-1},x^{*\prime}}$ satisfy an $r$-relation if choosing some $r$ rows of some $A(x',D')$ we have $A'(x',D')\vec{v}(x') = \vec{0}$. Here $\vec{v}$ denotes ${}^t(v_1,\cdots,v_m)$. Note that even if $v_1(x'),\cdots,v_m(x')$ satisfy an $r$-relation and another $s$-relation, it does not necessarily mean an $(r+s)$-relation.

We next define a classification of the characteristic roots. Let $\theta \in \{0,\pi\}$. Let

$$(x,\xi') \in \mathbf{R}^n \times \mathbf{R}^{n-1},\ x_1 \neq 0,\ \arg x_1 = \theta. \tag{5}$$

We define

$$\begin{aligned} M &= \{1,2,\cdots,m\}, \\ M_{0,\theta} &= \{\lambda \in M;\ \mathrm{Re}(x_1\varphi_\lambda(x,\xi')) = 0,\ \text{if } (x,\xi') \text{ satisfies (5)}\}, \\ M_{\pm,\theta} &= \{\lambda \in M;\ \pm\mathrm{Re}(x_1\varphi_\lambda(x,\xi')) > 0,\ \text{if } (x,\xi') \text{ satisfies (5)}\}, \\ M'_\theta &= M \setminus M_{0,\theta} \setminus M_{+,\theta} \setminus M_{-,\theta}\ . \end{aligned}$$

It is easy to see that $M_{0,\theta} \cup M_{+,\theta} \cup M_{-,\theta} \cup M'_\theta = M$ is a disjoint union.

Let $m_{0,\theta}$, $m_{\pm,\theta}$ be the number of the elements belonging to $M_{0,\theta}$, $M_{\pm.\theta}$, respectively. We assume that

$$M'_\theta = \emptyset,\ \forall\theta \in \{0,\pi\}. \tag{6}$$

We also need a condition for the microfumctions. Let

$$\begin{aligned} \omega(r) &= \{(x,\xi) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n;\ |x| < r,\ |\xi''| < r\,\mathrm{Im}\,\xi_n\}, \\ \omega'(r) &= \{(x',\xi') \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1};\ |x'| < r,\ |\xi'''| < r\,\mathrm{Im}\,\xi_n\}, \end{aligned}$$

and

$$\begin{aligned} \omega_0(r) &= \{(x,\xi) \in \omega(r);\ |x'| \leq r^{-1}|x_1|,\ |\xi''| \leq r^{-1}|x_1|\,\mathrm{Im}\,\xi_n\}, \\ \omega'_0(r) &= \{tx'^*;\ t > 0\}. \end{aligned}$$

We define

$$\mathcal{C}_0 = \varinjlim_{r>0} \Gamma_{\omega_0(r)}(\mathcal{C}_{\mathbf{R}^n},\ \omega_0(r)),$$
$$\mathcal{C}'_0 = \varinjlim_{r>0} \Gamma_{\omega'_0(r)}(\mathcal{C}_{\mathbf{R}^{n-1}},\ \omega'_0(r)).$$

Then we have the following

**Theorem.** *We assume* (1) – (3) *and* (6). *Let* $v_1(x'), \cdots, v_m(x') \in \mathcal{C}'_0$. *Then there exists an* $m_{+,0}$*-relation and an* $m_{+,\pi}$*-relation such that the Cauchy problem* (4) *has a solution* $u \in \mathcal{C}_0$ *if, and only if,* $v_1(x'), \cdots, v_m(x')$ *satisfy these relations.*

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

*Example 0 (Lewy-Mizohata operators).* If $P_\pm = D_1 \pm \sqrt{-1}x_1D_n$, then we have $M_{\pm,\theta} = \{1\}(= M)$, $M_{\mp,\theta} = \emptyset$. The above theorem means that $P_-u = 0$, $u(0, x') = v(x')$ is solvable for any $v \in \mathcal{C}'_0$ without any relations. In fact using the defining function we only need to let $u(x) = v(x''', x_n + \sqrt{-1}x_1^2/2)$. On the other hand, $P_+u = 0$, $u(0, x') = v(x')$ is solvable only for the case when $v(x')$ satisfies a one-relation. This means $v = 0$, and $u = 0$. It follows that $P_+u = 0 \Rightarrow u = 0$, i.e., $P_+$ is hypo-elliptic in $\mathcal{C}_0$ (See [6]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to $M_{+,\theta}$ cause obstruction, and correspondingly the Cauchy data must satisfy so many relations. Let us see the case $m = 2$.

*Example 1 (microhyperbolic operators).* Let $P(x, D) = D_1^2 - x_1^2D_n^2 + P'(x, D)$, $\operatorname{ord} P' \leq 1$. Without loss of generality, we may assume that $P'$ is a polynomial in $D_1$ of degree 1. Since $\varphi_1(x, \xi') = x_1\xi_n, \varphi_2(x, \xi') = -x_1\xi_n$, and $\arg \xi_n = \pi/2$, it is easy to see that $M_{0,\theta} = \{1, 2\}$, $M_{\pm,\theta} = \emptyset$ for $\theta \in \{0, \pi\}$. It follows that that (4) is solvable for arbitrary $v_1(x'), v_2(x') \in \mathcal{C}'_0$ without any relations (See [3]).

*Example 2 (Tricomi operators).* Let $P(x, D) = D_1^2 - x_1D_n^2 + P'(x, D)$, $\operatorname{ord} P' \leq 1$. We have $\varphi_1(x, \xi') = \sqrt{x_1}\xi_n$, $\varphi_2(x, \xi') = -\sqrt{x_1}\xi_n$. It follows that $M_{0,0} = \{1, 2\}$, $m_{+,0} = 0$, and that $M_{+,\pi} = \{1\}$, $M_{-,\pi} = \{2\}$, $m_{+,0} = 1$. It follows that there exists a 1-relation, and (4) is solvable if, and only if, the Cauchy data satisfy this relation. We can understand this phenomenon as follows. Let $\omega \subset \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ be a small neighborhood of $x^*$, and let $\omega^\theta = \{(x, \xi) \in \omega;\ x_1 \neq 0,\ \arg x_1 = \theta\}$, $\theta \in \{0, \pi\}$. At first we consider an elliptic boundary value problem in $\omega_\pi$, giving one boundary datum on $\{x_1 = 0\}$. Then we can always extend this solution to the hyperbolic region $\omega_0$. This case was considered also by [4]

*Example 3* (hypoelliptic operators). Let $P(x, D) = D_1^2 + x_1^2D_n^2 + P'(x, D)$, $\operatorname{ord} P' \leq 1$. Since $\varphi_1(x, \xi') = \sqrt{-1}x_1\xi_n, \varphi_2(x, \xi') = -\sqrt{-1}x_1\xi_n$, it is easy to see that $M_{-,\theta} = \{1\}$, $M_{+,\theta} = \{2\}$, $m_{+,\theta} = 1$ for $\theta \in \{0, \pi\}$. There exist an $m_{+,0}$-relation and an $m_{+,\pi}$-relation such that the Cauchy problem (4) uniquely has a solution $u \in \mathcal{C}_0$ if, and only if, $v_1(x'), \cdots, v_m(x') \in \mathcal{C}'_0$ satisfy both of these relations. In most cases two 1-relations mean a 2-relation, but this is not always true. If this is true (4) is solvable only in the case $v_1 = v_2 = 0$, and $u = 0$. In other words, $Pu = 0$ does not have any non-trivial solutions. It is well-known that this is true if the principal symbol $\sigma_1(P')$ of the lower order term satisfies $\xi_n^{-1}\sigma_1(P') \notin \{\sqrt{-1},\ \sqrt{-13},\ \sqrt{-15}, \cdots\}$(See [2,5]). Of course our result applies for higher order operators, too.

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